Timed Strategic Games

A new game theory for managing strategic plans in the time dimension

Salvatore Alessandro Sarcia'
OnESE.org Forum
Independent Research Division
sarcia@onese.org

***Abstract— Background*: Over the last 60+ years a number of wars of different kinds has been taken place worldwide. Scholars refer to the latest ones as 4th Generation wars (or warfare) where the confrontation is no longer between state actors, but between non-state actors (mainly guerrilla/insurgents) and state actors. *Aims*: Since the old strategic game theory (e.g. Prisoner's Dilemma) is not able to explain 4th Generation wars because of asymmetry and the lack of a timed framework, we define a variation of this theory that we call Timed Strategic Game with the aim of bringing up the discussion on what it is really needed to plan and manage military campaigns such as Afghanistan, Iraq, Libya, Somalia, Lebanon, Israel-Palestine, and etc. *Method*: The theory being defined deals with asymmetry and includes a temporal dimension. Our definition is completely new. We did not use an automaton approach as usually done in game theory. We considered a continuous parametric function varying based upon the variable time. The use of the variable time in strategic games is the main novelty of this work. *Results*: Based upon the definition of "Timed Strategic Game", we put forward the definition of "Timed Prisoner's Dilemma" and another one that we called IN-OUT game, which was set up specifically to model the main features of 4th Generation wars. Furthermore, the contribution of this work is also with the definition of Stable and Unstable Timed Nash Equilibria, which are the extension of "Nash Equilibria" to Timed Strategic Games. *Conclusions*: In this research we show a way of modeling asymmetric strategic games over time. Even though the suitability of the application of this new theory has to be tried out in practical terms, it is the first step to have a theoretical framework where strategic games can finally be categorized in terms of time. Additional contributions are that the proposed model is general purpose regardless of 4th Generation wars; we observed that Timed Strategic Games could also help model value-based management, earned value analysis, and share-market dynamics.**
Keywords—game theory; asymmetric war; asymmetric games, timed Nash equilibrium; stable Nash equilibrium; unstable Nash equilibrium; 4th Generation wars.

I. INTRODUCTION

This paper addresses from a mathematical point of view, one of the most severe problems that have always plagued the humankind, i.e., the war. Throughout the 20th and 21st centuries, what helped rulers, from a theoretical point of view, make the right decisions at the right time has been game theory [1], [12], [13]. Literature is extremely reach on using game theory in defense and security areas at the time of world and cold wars, but it is completely poor or even absent on using game theory on 4th Generation wars (the latest). No one has managed to state a suitable theory to try to explain, analytically, 4th Gen wars. To the best of our knowledge, this work on 4th Gen wars is the first attempt to this end.

The effort that organizations and governments have made so far to deal with all kinds of war is only about technology, mainly, software and systems (S&S). And, this is particularly positive not only at military level, but also for civilians, for instance against diseases, natural disasters, and resource exploitation. The old game theory on symmetric wars cannot help anymore. Cyber defense is the most representative example on the current situation where asymmetry makes a difference [11]. Few people can attack a whole country.

It is indisputable that technology has dramatically improved military (weapons, control systems etc.) and civil (disaster recovery, medical science etc.) power. For instance, improving awareness over a number of situations such as warfare, homeland security, natural disaster, and transportation control is a way of increasing the power of a country. Software development is the basis of it, and, perhaps one of the largest power-driver worldwide [10], [3]. Other approaches to increase power are on how to measure [16], [18] and retrieve [15], [2] information, classify people [4], [19], [7], reduce failures [6], and analyze risks automatically [17].

The point is that technology is not enough to help deal with 4th Gen wars. We need a theory, which can model this particular phenomenon with all its characteristics. An analytical model can be taken as a means to which all variations of the phenomenon can be linked and thus investigated. Without an explicative model, rulers may have the strongest power ever, but will not make the right decisions (e.g., on planning and managing military operations). The proposed model was devised to be complementary to the research on situation awareness and S&S. A suitable model of 4th Gen wars, as the one we propose here, should be able, at least, to represent the following characteristics, which were inferred by the author from [9] and [20]: 1) asymmetry of opponents in terms of forces, resources, warfare, and strategies, 2) the duration of the confrontation lasting forever (i.e., inability to end the confrontation), 3) low utility of having military power, in fact, the more the confrontation lasts, the more insurgents gain, 4) the fact that any strategy/action of organized actors (e.g., state-actors, GOs, NGOs) returns some benefits right after its application, but it inexorably fades away as time goes on, while since insurgent-like actors do not have institutionalized

S. A. Sarcia' is with the Italian Ministry of Defense – Army General Staff, Rome 00187, Italy (e-mail: sarcia@onese.org). This material was partially supported by OnESE.org Forum – Independent Research Division, 01028 ORTE (VT), Italy. The views and conclusions contained in this document are those of the author and should not be interpreted as representing the official policies, either expressed or implied, of the Italian Ministry of Defense - Army General Staff.

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strategies, their payoff is generally low, but is almost constant over time.

The analytical model that we define here is completely general regardless of 4th Gen wars. Consequently, the model is neither right nor wrong. The point is that, whether or not the model can fit all the characteristics of 4th Gen wars and explain this dreadful phenomenon. What is important to bear in mind, however, is that any mathematical model, this included, can only explain partially a phenomenon. The model defined here is an attempt to represent mathematically the principal characteristics of 4th Gen wars; therefore, its utility has to be evaluated in the field. An empirical evaluation of the model is out of the aim of this work.

The main contribution of this paper is that we defined a timed variation of strategic game theory that we called Timed Strategic Game. However, our definition does not use an automaton approach. We considered a continuous parametric function of time. Additional contributions of this work are: the definition of a game called "Timed Prisoner's Dilemma" and the one that models 4th Gen wars, called IN-OUT game. We also defined a Timed Nash Equilibrium, which is the temporal extension of the Nash Equilibrium, and the concept of stable, partially left/right-handed stable, and unstable timed Nash Equilibrium.

This paper is organized as follows. In section II we define Timed Strategic Games from a mathematical point of view. In Section III we discuss repeated Timed Strategic Games. In Section IV we give additional insights on the importance of this new theory. In section V we provide some definition and a theorem on stability, instability, and partially left/right handed stability of Timed Nash Equilibrium. Section VI refers to the definition of the game called Timed Prisoner's Dilemma. Section VII is on the definition of the game called IN-OUT, which is the game that explains either 4th Gen wars or similar phenomena such as value-based management, share-market dynamics, and earned value analysis. The paper ends with some final remarks and future work.

II. Timed Strategic Games

The easiest form of a strategic game is usually presented with two players (called player 1 and player 2) and four actions (A_1, A_2, A_3, A_4) available to players, e.g. A_1, A_2 available to player 1 (row) and A_3, A_4 available to player 2 (column). Sometimes it can happen that $A_1 \equiv A_3$ and $A_2 \equiv A_4$. There is also a payoff function that maps actions to (real) values. Formally, a strategic game consists of [14]:

- A finite set of players (N)
- A finite set of actions available to each player, let i be a generic player with $i \in N$, then A_i is a nonempty set of actions available to player i
- A payoff function (also called utility function) u_i: $A \rightarrow \Re$, which maps actions to real numbers.

The utility function u is a preference relation, which only depends on A and $\Re$. When dealing with two-player strategic games, u can be represented by a table as in Figure 1 [13].

In strategic games the actions are taken independently, simultaneously and each player knows the actions of the others. However, the player does not know which action the opponent will take. Players are assumed to have a rational behavior, which means that a player is fully aware of his alternatives, has well-defined preferences, and makes his choices after a decision-making process with the aim of increasing his payoff (utility).

	A_3	A_4
A_1	v_1, v_2	v_3, v_4
A_2	v_5, v_6	v_7, v_8

Figure 1. Representation of a two-player strategic game where each player has two different actions available.

With respect to Figure 1, the individual figure in the table is a real number by definition. Note that, those numbers are constant once defined. Let us now change this definition stating that those numbers can change over time. To introduce the concept of time, we do not use the automaton theory as usually done in the field of game theory. We consider a continuous function that changes based upon the variable time. Formally, the strategic game being defined, that we call a Timed Strategic Game, is as follows:

- A finite set of players (N)
- A finite set of actions available to each player, let i be a generic player with $i \in N$, then A_i is a nonempty set of actions available to player i
- A payoff continuous function (that we call timed-utility function or timed-payoff function) u_i: $(A, T) \rightarrow \Re$, which maps actions and times to real numbers with ($T \subseteq \Re^+ \cup \{0\}$).

This definition is new with respect to those that have been defined lately [8]. Note that, we will use the following notation to denote positive real numbers with the singleton 0, i.e., $\Re_0^+ \equiv \Re^+ \cup \{0\}$. From a geometrical point of view, since set A of actions cannot be represented on a unique Cartesian plane, we define a function u_i: $T \rightarrow \Re$ for each element of set A, i.e. for each element of A we have a function $u_i(k; \theta; t) = r$, with $r \in \Re$ and k and θ two parameters specifying u_i. We denote the function as $u_i^A(t) = r$. To keep our notation as simple as possible, we will use $u_i^A(k; \theta; t)$ instead of $u_i(A, k; \theta; t) \equiv u_i(a, b, c, \ldots, k; \theta; t)$ or more easily $u_i^A(t)$ when omitting the two parameters does not create confusion. We will use $u_i^{a,b,c,\cdots}(k; \theta; t)$ or $u_i^{a,b,c,\cdots}(t)$ to specify the nature of the mapping between actions. A convenient function that we employ to define $u_i^A(t)$ is as follows:

$$u_i^A(k;\theta;t) = \int_0^t v^A(k;\theta;t)\,dt = \int_0^t \frac{k}{e^{\frac{t}{\theta}}}\,t\,dt \qquad (1)$$

with $k \in \Re$, $\theta \in \Re^{+}$, are two parameters shaping the function $v^{A}(k;\theta;t)$ and t is the variable time. Since k and θ are two parameters as defined above, $u_i^{A}(k;\theta;t)$ can assume any value based upon the values of k and θ. This is the reason why equation (1) is not arbitrarily stated. Having a sufficient number of observations, one can calibrate the two parameters (i.e., k and θ), selecting the most suitable function. Note that, k regulates the highness of v, θ is responsible for the wideness of v, and e is the exponential. Note that, k and θ have to be treated as constants. Since $u_i^{A}(k;\theta;t)$ is completely specified by $v^{A}(k;\theta;t)$, it is important noting that, $v^{A}(k;\theta;t)$ denotes the right-side mathematical function in (1). Sometimes we omit the apex A to simplify the notation. As an example, let us state that k=2 and θ=4, then the payoff function is as follows:

$$u_i^{A}(k=2;\theta=4;t) = \int_0^t v^{A}(k=2;\theta=4;t)dt = \int_0^t \frac{2}{e^{\frac{t}{4}}} t dt. \quad (2)$$

Let us now consider Figure 1 again. We state that each value v_j, for j = 1 to 8, is provided as follows:

$$v_j(k^j;\theta^j;t) = u_i^{A}(k^j;\theta^j;t) = \int_0^t \frac{k^j}{e^{\frac{t}{\theta^j}}} t\, dt. \quad (3)$$

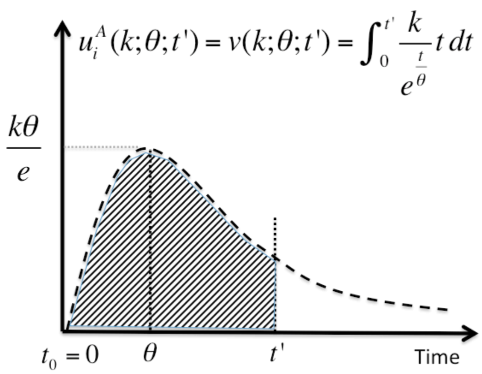

Figure 2. The area representing the payoff v in Figure 1 for $t = t'$.

Note that, apart from A, equation (3) is only function of the time t since k^j and θ^j are two known parameters. Once we fix a value for t, e.g. $t = t'$, equation (3) provides a real value which represents the area in Figure 2. It is important noting that, this particular shape was chosen to represent the characteristic that any military strategy/action returns some benefits right after its application, but it inexorably fades away as time goes on. The theory is completely valid even if we choose a different shape for the function in equation (1) and Figure 2.

The payoff table in Figure 1 becomes as in Figure 3. Generally speaking, each function $v_j(k^j;\theta^j;t)$ for j = 1 to 8 is different from each other because of parameters k^j and θ^j. The representation in Figure 2 assumes that k is nonnegative. However, k can be negative as well, and the curve would be drawn symmetrically below the x-axis. Note that, when t goes to infinity the integral in (3) would provide its maximum, i.e. all the area below the function $v(k;\theta;t)$.

It is important noting that, if the notion of the time does not matter the timed-strategic game thus defined can be treated as a normal strategic game. That is, the payoff function would provide the total area in Figure 2 as t tends to infinity. This means that the normal (usual) setting can be considered as a specific case of a timed strategic game (long term setting).

$t=t'$	A_3	A_4
A_1	$v_1(k^1;\theta^1;t'), v_2(k^2;\theta^2;t')$	$v_3(k^3;\theta^3;t'), v_4(k^4;\theta^4;t')$
A_2	$v_5(k^5;\theta^5;t'), v_6(k^6;\theta^6;t')$	$v_7(k^7;\theta^7;t'), v_8(k^8;\theta^8;t')$

Figure 3. Representation of a timed-payoff function for a two-player strategic game for $t = t'$.

The function depicted in Figure 2 has some interesting features. It is smooth and continuous, arg $\max_t u_i^{A}(k;\theta;t) = \theta$, which is a singleton, and its maximum can be calculated by $(k\bullet\theta)/e$. The total area below the curve is $k\bullet\theta^{2}$ and the part being on the left of the vertical line $t = \theta$ is always 26.42% of the total as well as the remaining area (on the right) is always 73.58% of the total. The reason why equation (1) is a convenient function is because of all these characteristics.

$t \rightarrow +\inf$	A_3	A_4
A_1	$k^1(\theta^1)^2, k^2(\theta^2)^2$	$k^3(\theta^3)^2, k^4(\theta^4)^2$
A_2	$k^5(\theta^5)^2, k^6(\theta^6)^2$	$k^7(\theta^7)^2, k^8(\theta^8)^2$

Figure 4. Payoff function for t that tends to infinity. The result $k\theta^2$ is the result of calculating the integral in (2). The apex "2" applied to the parentheses means power 2.

Note that, if the time does not matter or we consider an infinite time, the payoff function in Figure 3 becomes as in Figure 4

III. REPEATED TIMED STRATEGIC GAMES

The game thus defined has the potential of including repeated strategic games, as well. A repeated strategic game is a strategic game, which is played a number of times with the same setting as for a (nonrepeated) strategic game [21]. In addition, players can take their actions based upon recording the history of the game. It would be the experience gathered by playing the game over time. However, repeated games have not an explicit time-related setting, which instead is only hinted.

Let us define a repeated timed strategic game based upon the setting of a timed strategic game as defined above. Let us assume that player 1 has two actions (A_1 and A_2) available as in Figure 3. Then, a repeated strategic game would be as in Figure 5. Note that, functions in Figure 5 may be different from each other when having different parameters (k, θ). From $t = t_0$ to $t = t_1$ player 1 gets his payoff based upon the function $u_i^{A}(k^1;\theta^1;t_1)$. Note that, the gain stemming from $u_i^{A}(k^1;\theta^1;t)$ is interrupted by Action 2 even though player 1 gets his payoff from $u_i^{A}(k^2;\theta^2;t)$. Therefore, a repeated timed strategic game differently from a repeated strategic game takes into account the notion of time related to the time when an action is taken and how long an action lasts.

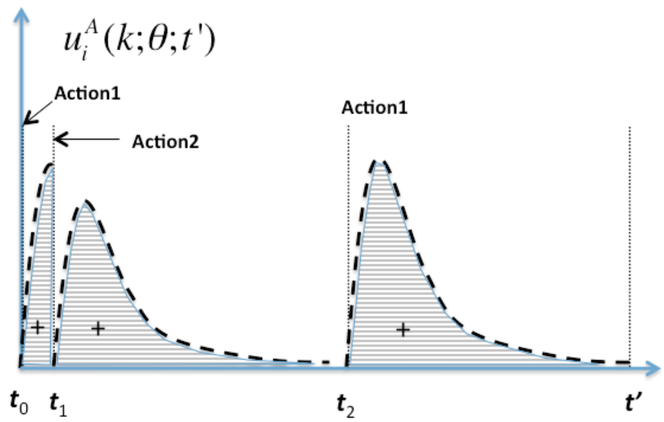

Figure 5. A repeated timed strategic game payoff functions.

At $t = t'$ we can compute the entire payoff gained from the game by adding the individual contributions from t_0 to t'. We can compute the payoff for player 2 as well and checking who would be the winner. It is important noting that based upon this definition of the repeated timed strategic game we now have the opportunity to see how the payoff stemming from the actions taken by the players unfolds over time.

IV. Interpretation of Timed Strategic Games

In this section we explain why a timed strategic game is important as well as the reason why we thought of defining it. First, there are a lot of situations where the setting without the notion of time is not realistic. We specifically refer to complex situations as, for instance, military campaigns. In reality, when a party (a player in our case) takes an action against the other, his payoff arising from that action is not instantaneous as in usual strategic games (which is unreal anyway). The point is that, even though we know the payoff stemming from a combination of actions, the important thing is to know the given payoff with respect to the variable time. For instance, what would be best between (a) a price of \$10 per year for 5 hundred years and (b) \$50 per year for 1 hundred years? The answer is not self-evident. Since the total is always \$5,000 the only difference is upon the time when money is given. Therefore, the best choice would depend upon the need of the players' situation over time in terms of cumulative amount.

As for military campaigns, the variable time is vital and a game that does not take it into account can only explain partially the expected results. Even though we are not referring to decision-making problems as for the examples (a) and (b), the point is that a strategic game with the notion of time would be much more useful than its homologous without temporization.

A timed strategic game is the setting to explain not only what to do to win the game as usual, but also how the payoff function will progress over time giving more insights on the evolution of the game for prediction purposes.

V. Nash Equilibrium in Timed Strategic Games

Stability is a concept that is not new to the field [5] and the definition of a Nash Equilibrium [12] is absolutely general to be molded to different interpretations. Let $\Re_0^+ \equiv \Re^+ \cup \{0\}$. We state the following theorem and definitions:

Theorem 1. Let G^T = <N, (A_i), $(u_i^A(k;\ \theta;\ t))$> be a timed strategic game, then for $t = t'$ for each $t' \in \Re_0^+$, G^T is a strategic game in the form <N, (A_i), (u_i)>.

Proof. To prove the theorem we have to show that $(u_i^A(k;\ \theta;\ t{=}t')$ includes (at least) the same mapping as the one done by function u_i: A→$\Re$ for $i \in$ N, i.e., u_i be a utility function for the game <N, (A_i), (u_i)>. Since u_i^A maps (A, $\Re_0^+$)→$\Re$, it is also true, by construction, that u_i^A maps A→$\Re$ for each $i \in$ N. We have now to prove that the output of $u_i^A(k;\ \theta;\ t{=}t')$ for each $t' \in \Re_0^+$ is a value $r \in \Re$. This statement is true by definition because once we fix the values for k, θ, and t (i.e., $t = t'$) the integral in (3) is a definite integral providing a real number as defined for integrals if the integral exists. However, since the function inside the integral in (3) is smooth and continuous, we can conclude that every definite integral of the function exists. If k is negative the integral in (3) is negative, as well. The meaning of a negative integral is that the area would be on the negative side of y-axis. Δ

Definition 1. Let <N, (A_i), $(u_i^A(k;\ \theta;\ t))$> be a timed strategic game, then we define to be a Timed Nash Equilibrium for $t = t'$ a profile a* ∈ A of actions such that for each player $i \in$ N we have $u_i^{a^*_{-1},\, a^*_1}(k;\ \theta;\ t = t') > u_i^{a^*_{-1},\, a_1}(k;\ \theta;\ t = t')$, or using an equivalent notation $u_i(a^*_{-1};\ a^*_1;\ k;\ \theta;\ t = t') > u_i(a^*_{-1};\ a_1;\ k;\ \theta;\ t = t')$. This definition underlines that a Timed Nash Equilibrium for $t = t'$ is as such at point $t = t'$, but for different values of t the Equilibrium may not hold anymore.

Definition 2. We define to be a *Stable Timed Nash Equilibrium* for $t = t'$ a Timed Nash Equilibrium for $t = t'$ such that it does not exist any value t'' for each $t'' \in \Re_0^+$ that turns the Timed Nash Equilibrium for $t = t'$ into a non Timed Nash Equilibrium at point $t = t''$.

Definition 3. We define to be a *Partially right-handed Stable Timed Nash Equilibrium* for $t > t'$ a Timed Nash Equilibrium for $t = t'$ such that it does not exist any value $t'' > t'$ with $t'' \in \Re_0^+$ that turns the Timed Nash Equilibrium for $t = t'$ into a non Timed Nash Equilibrium at point $t = t''$.

Definition 4. We define to be a *Partially left-handed Stable Timed Nash Equilibrium* for $0 < t < t'$ a Timed Nash Equilibrium for $t = t'$ such that it does not exist any value $0 < t'' < t'$ with $t'' \in \Re_0^+$ that turns the Timed Nash Equilibrium for $t = t'$ into a non Timed Nash Equilibrium at point $t = t''$.

Definition 5. We define to be an *Unstable Timed Nash Equilibrium* for $t = t'$ a Timed Nash Equilibrium for $t = t'$ if it exists a couple of value t'' and t''' for each $t'', t''' \in \Re_0^+$ such that $t'' < t' < t'''$. In other words, it is always possible to arbitrarily fix two values $t'', t''' \in \Re_0^+$ such that $t'' < t' < t'''$.

VI. Timed Prisoner's Dilemma

Based upon the definitions above, we now define a *Timed Prisoner's Dilemma* game as an extension of the usual Prisoner's Dilemma game (Figure 6). The setting of the timed game is the same as the usual one apart from the payoff function that is defined over the variable time t. With respect to Figure 6, the condition for the game to be in the form of Prisoner's Dilemma is that $v_2 > v_1 > v_4 > v_3$. As for the iterative setting of the game, there is an additional condition, i.e., $2v_1 > v_2 + v_3$. Note that, actions A_1 and A_2 are both available to the two players. Payoff functions are as in Table1.

	A_1	A_2
A_1	v_1, v_1	v_3, v_2
A_2	v_2, v_3	v_4, v_4

Figure 6. Payoff function of the game Prisoner's Dilemma.

Note that, based upon values in Table 1 we see that when t→+inf (or for a sufficiently large value of t) the game in Figure 6 is in the form of Prisoner's Dilemma because the condition that $v_2 > v_1 > v_4 > v_3$ holds, being 32 > 24 > 16 > 8 and the iterative condition that $2v_1 > v_2 + v_3$ is satisfied because (2•24) > (32 + 8), i.e., 48 > 40.

Table 1.Setting of generalized timed Prisoner's Dilemma.

j	v_j	k^j	θ^j	*Area for t = 3*	*Total area* $(k\bullet\theta^2)$
1	v_1	7.4	1.8	11.9	24
2	v_2	2	4	5.5	32
3	v_3	5.5	1.2	5.7	8
4	v_4	25	0.8	14.2	16

The payoff function of timed Prisoner's dilemma for t→+inf is as in Figure 7. The four-payoff functions v_1, v_2, v_3, and v_4 are depicted in Figure 8.

t→+inf	Cooperate	Defect
Cooperate	*24, 24*	*8, 32*
Defect	*32, 8*	*16, 16*

Figure 7. Payoff functions of timed Prisoner's Dilemma

Each function has a different shape as specified in Table 1 by parameters (k^j, θ^j) and depicted in Figure 8.

The interesting point is that, for a sufficient large value of t, e.g. t→+inf, the only Nash Equilibrium is for the combination (Defect, Defect), i.e. both players decide to Defect. This is not an unstable Nash Equilibrium, however. It is a stable Nash Equilibrium as usually defined for Prisoner's Dilemma because of the fact that time t is sufficiently long to make the game stable.

Let us now consider $t' = 3$ and calculate the area below each curve in Figure 8. The results are shown in Table 1 (*Area for t = 3*). Therefore at $t' = 3$ the payoff function is as in Figure 9. Note that, the payoff function in Figure 9 does not refer to Prisoner's Dilemma because the condition $v_2 > v_1 > v_4 > v_3$ does not hold. However, in Figure 9 there are two Nash Equilibria (Cooperate, Cooperate) and (Defect, Defect). Based upon the defined timed strategic game, the prisoners' situation changes over time. However, the two timed Nash Equilibria are unstable because Equilibria change according to the time.

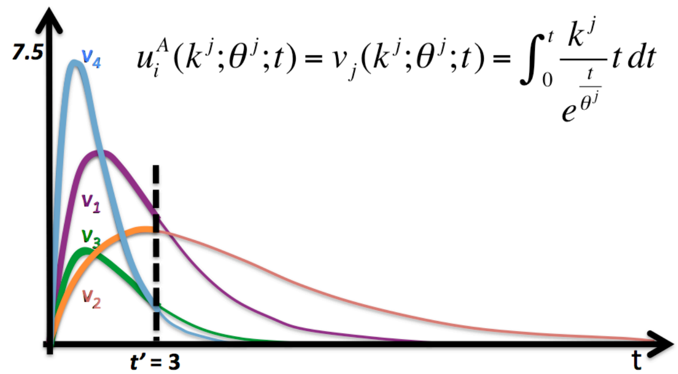

Figure 8. A geometrical representation of the payoff functions of timed Prisoner's Dilemma.

Due to space limitation, we do not show what would happen to the payoff function and then to timed Nash Equilibria for additional values of t such as $t = 6$, $t = 9$, $t = 12$ etc.

t = 3	Cooperate	Defect
Cooperate	*11.9, 11.9*	*5.7, 5.5*
Defect	*5.5, 5.7*	*14.2, 14.2*

Figure 9. Payoff function of timed prisoner's Dilemma for $t = 3$.

VII. USING TIMED STRATEGIC GAMES IN 4TH GEN WARS.

As stated in Section "*Interpretation of Timed Strategic Games*", the Game Theory being proposed was set up to interpret a variety of phenomena such as *4th Gen Wars* [9] also called *Asymmetric Wars* or *Wars Amongst the People*" [20], value-based management, earned value analysis, and share market. Because of space limitations, we will only focus upon 4th Gen Wars providing a real example of how the proposed theory would be usable in practical terms for planning and managing purposes.

Since this work is not a political treaty, our terminology would need to be better specified, we prefer explaining what we mean by "4th Gen wars" through some examples. We refer to all conflicts/confrontations that are similar in nature to military campaigns such as the ones in Iraq, Afghanistan (i.e. so called war on terror of the USA-UK, NATO since 2001 on, but also the one that Russian forces waged in 1979-89), Somalia, Libya, Lebanon, Israel-Palestine, etc. Low intensity, asymmetry, guerrilla, insurgency, and non-state actors characterize these kinds of wars.

From a game theory point of view, in this section, we show that the proposed model is able to represent all characteristics of 4th Gen wars stated in the "Introduction", i.e., 1) asymmetry, 2) indefinite duration, 3) partial uselessness of military power, and 4) payoff returned from the application of any strategy/action of organized actors (e.g., state-actors, GOs, NGOs) increases in the beginning and decreases over time, while the payoff function of insurgent-like actors is generally very low, but is almost constant over time.

Without loss of generality, we intentionally simplify the game by considering two agents that we call player 1 (row) and player 2 (column). These two players take part in a timed strategic game classified as asymmetric. For the sake of clearness we consider neither repeated nor Bayesian games. We left them as future work. Our asymmetric game consists of two non-interchangeable types of player that we call identities. These two identities are not coalitions, however. They represent the unique nature of the players belonging to the considered group.

As a consequence of that, a player can belong to one and only one identity. The concept of non-interchangeability is that the identity of one of the players cannot be replaced by the identity of the other. This asymmetry characterizes the payoff function, as well. We refer to the identity "insurgents" as player 1 (row) and to identity "state actor" or the like as player 2 (column).

There are two different actions $A_i = (A_1, A_2)$ available to both players. We consider A_i as strategies in political-military terms. In particular, A_1 is the action of taking part in the asymmetrical confrontation against the opponent (IN) and A_2 represents the action of getting/being out of the game (OUT).

It is important noting that at a theoretical level when a player is out of the game he is no longer a player and so he should not be part of it. However, the point that we want to remarkably make is that the action of being out of the game is part of the game and then we treat such a player as a player who is not playing, but he may eventually participate in it again (playing again).

$t = t'$	IN	OUT
IN	α, β (WAR)	$0, -\beta$ (CRISIS-TURMOIL)
OUT	$-\alpha, \beta$ (CRISIS-OCCUPATION)	$0, 0$ (PEACE)

Figure 10. Payoff function for "4th Gen War"-strategic game.

A generalized payoff function for $t = t'$ is depicted in Figure 10. That payoff function says that when both players are IN the players are confronting each other. We call it WAR. When they are both out of the game, no one is gaining anything. We call it PEACE. The other two combinations are situations of CRISIS (i.e., CRISIS-TURMOIL and CRISIS-OCCUPATION). Note that, in reality the combination (OUT-IN, i.e. occupation) is far from happening in 4th Gen wars. However, this is what happened with the USA troops in Afghanistan and Iraq. More likely to happen is the combination IN-OUT (i.e. turmoil), when insurgents (player 1) take over the crisis area and the state-actor/alliance is still to come.

In IN-OUT (i.e., turmoil) player 1 does not gain anything and player 2 is loosing $-\beta$, therefore player 2 is facing a situation where he has to make a decision whether it is more convenient to stay out of the game to make player 1 gain zero but loosing $-\beta$ or to get in to gain β but making player 1 gain α.

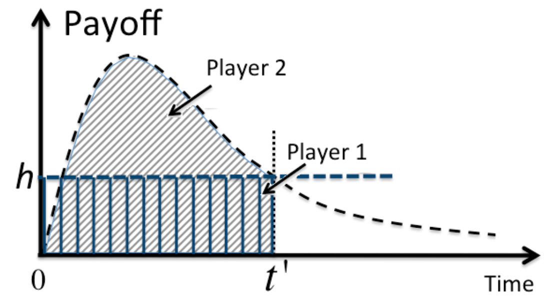

Figure 11. IN-IN payoff functions.

As far as OUT-IN (occupation) is concerned, player 1 looses $-\alpha$ while player 2 gains β. OUT-IN is not convenient for player 1. This is the formal representation that player 1, when the geographical area where he has his interests is occupied, tends to get into the game. The proposed game can also model that player 1 gains his payoff as long as player 2 is in the game, and that player 1 tends to keep player 2 playing.

Let us now see the setting for values $\alpha, \beta \in \Re$. Let us consider now, in Figure 11, areas:

- $\alpha = h \bullet t'$, region colored with vertical lines,
- $\beta = \int_0^{t'} \frac{k}{e^{\frac{t}{\theta}}} t \, dt$ region colored with transversal lines.

There is a substantial difference between these two timed payoff functions (i.e., the two areas) in Figure 11. Player 1's payoff function increases linearly over time. Player 2's payoff function (area) increases in the beginning and then increases progressively less to zero, which is the behavior observed for 4th Gen wars. Based upon the defined Timed Strategic Game, we consider the situation at time $t = t'$ when both α and β are constant (i.e., as a snapshot for $t = t'$). The asymmetry of the game is just in this difference. The more t goes on, the more player 1's payoff linearly increases, and the less player 2's payoff increases (at infinity the increment is zero). Then, for a sufficient long period of time t', player 1's payoff will always be greater than player 2's payoff if $h > 0$. It is worth noting that the proposed model can represent the nature of player 1 such that he gains his payoff due to the only fact that player 2 is playing (IN) as well as the model can represent the situation that if player 2 were OUT, player 1 would not gain anything.

Differently from player 1, to gain his payoff, player 2 has to take the right actions. In summary, the proposed game (Figure 10) recreates the situation occurring for 4th Gen wars that the only way that player 2 has to decrease (or wipe out) player 1's payoff is to get out of the game. Conversely, in order for player 2 to increase his payoff the only way is to be part of the game (IN-OUT dilemma). Player 1 does not find convenient to stay out of the game when player 1 is IN, because of the negative score ($-\alpha$). The combination OUT-IN is modeled to be unlikely to happen. The combination IN-OUT explains the situation when insurgents have taken over an area, which becomes an area of crisis, and a state-actor (i.e., player 2) intervention is needed. Until the state-actor intervenes (he is OUT), his payoff is negative ($-\beta$).

t = 10 ys	IN	OUT
IN	2, 4 (WAR)	0, -4 (CRISIS-TURMOIL)
OUT	-2, 4 (CRISIS-OCCUPATION)	0, 0 (PEACE)

Figure 12. Numerical example on the "4th Gen War" game

Note that, in Figure 10 there are two Timed Nash Equilibria: one for IN-IN (WAR) and one for OUT-OUT (PEACE). So, when both players are IN, i.e. at WAR, they tend to remain there (i.e., the characteristic of 4th Gen-wars that last forever). However, when players are not in the game, they can stay in PEACE forever, unless external events (not part of the game) take place.

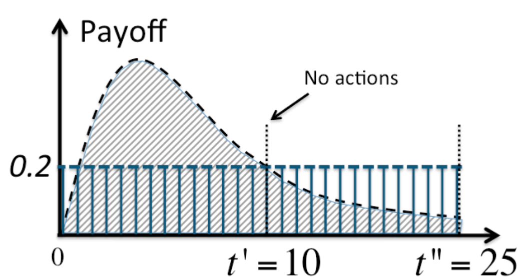

Figure 13. Situation at time *t''* where player 1 gets a dominant strategy.

It is now clear enough that this model represents the situation that the only strategy to stop WAR is to act unilaterally (i.e., irrationally) getting out of the game (OUT) and, if in PEACE, do not act unilaterally keeping the state.

If the proposed model were a suitable model of the reality, we can use it to state the right time to get out of the game, i.e., as long as player 2 has a dominant strategy. To do so, we should know the payoff functions by statistically estimating *k, θ, and h*. Figure 12 and 13 show an example of the game thus defined (IN-IN) assuming to know *k, θ, and h*, that is:

- $t' = 10$, $\alpha = 2$, $h = 0.2$
- $\beta = \int_0^{t'} \frac{k}{e^{\frac{t}{\theta}}} t\, dt \cong 4$ with $t' = 10$, $k = 4$ and $\theta = 1$.

At $t = t' = 10$ the game is in favor of player 2 who is going to win the game since he has a higher score. At time $t' = 10$, neither player 1 nor player 2 did anything because of the nature of the confrontation (Nash Equilibrium). Player 1 is not leaving the area of the conflict staying IN. Based upon the defined game the situation gets worse for player 2 as *t* goes on (Figure 13). Let us calculate the payoff at $t'' = 25$:

- $t'' = 25$, $h = 0.2$ then $\alpha = 25 \bullet 0.2 = 5$
- $\beta = \int_0^{t''} \frac{k}{e^{\frac{t}{\theta}}} t\, dt \cong 4$ with $t'' = 25$, $k = 4$ and $\theta = 1$.

The payoff matrix is as follows:

t = 25 ys	IN	OUT
IN	5, 4 (WAR)	0, -4 (CRISIS-TURMOIL)
OUT	-5, 4 (CRISIS-OCCUPATION)	0, 0 (PEACE)

Figure 14. Evolution of the matrix in Figure 12 after 25 ys.

As we see in Figure 14, even if the two Timed Nash Equilibria are preserved, the situation is no longer in favor of player 2.
In summary, the proposed game is a theoretical model, which is able to represent the situation as follows:

- The asymmetric game is, after a sufficient period of time, in favor of insurgents (player 1) and against the state-actor/alliance (player 2) even if the latter has, in the beginning, a more "profitable" strategy
- For player 2, the only way of breaking down the unlucky situation is to get out of the game even if the behavior classified as irrational and unilateral
- When planning a military campaign, the state actor (e.g., player 2) must identify in advance the correct time of getting out of the game (exit strategy) otherwise, as analytically explained above, insurgents (player 1) will have available the winning (strictly dominant) strategy
- From the side of the state actor, the only winning (strictly dominant) strategy is to avoid that crises may happen; otherwise, if crises take place all potential players will be at stake.

Let us now consider the graph in Figure 15. It shows the dynamics of the game. The starting point is PEACE where both player 1 and player 2 are OUT-OUT of the game. The only two suitable cases that we can expect from PEACE are TURMOIL, if player 1 comes IN, and OCCUPATION, if player 2 comes IN. However, since PEACE is a Timed Nash Equilibrium, we assume that both transitions happen because of external causes or the behavior of the players is not rational. We identify these "*irrational*" transitions by dashed arrows in Figure 15. Once the game is in the state TURMOIL, if player 2 comes IN (i.e., the state actor intervenes) the state of the game turns into WAR otherwise, if player 1 comes OUT, the state of the game turns into PEACE.

If, for some reasons, player 1 decides to come out of the game, the game can turn into PEACE again, which is a steady state and determined by the strategy profile OUT-OUT that is a Timed Nash Equilibrium. However, the state of the game can turn into WAR, if player 2 intervenes, i.e., he comes IN. Since WAR is a steady state determined by the strategy profile IN-IN that is a Timed Nash Equilibrium, to avoid WAR, player 2 should behave irrationally, i.e., he should come OUT of the game (i.e., change his action unilaterally). With respect to OCCUPATION, it is important noting that the intervention (OUT-IN) is the most wrong choice that player 2 can do if the aim is to maintain PEACE.

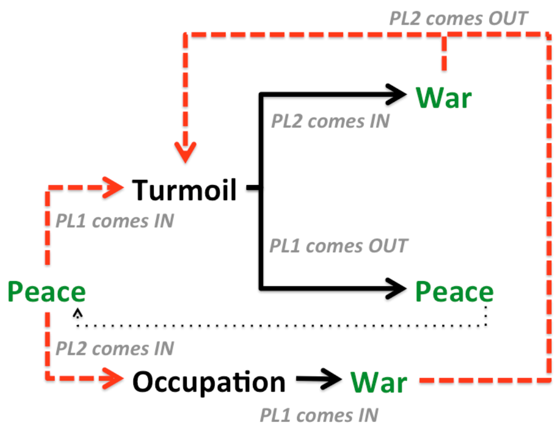

Figure 15. Dynamics of the game IN-OUT

If the proposed model represents the reality, when dealing with real 4th Gen Wars the strategy to maintain peace should be to prevent crises by avoiding both occupations and turmoil. Whenever a state actor/alliance (player 2) decides to manage a crisis (to be IN), the timing of the operation should be clearly defined through the suggested model to avoid the loosing situation shown in Figure 13, i.e., getting out as far as it is convenient. If player 2 is stuck in a war, anyway, the only possible action is to get out of the game, wiping out player 1's payoff and investing resources to take player 1 out of the game.

VIII. CONCLUSION

In this paper we defined a new theory concerning strategic games. The novelty is that we introduced a temporal dimension in the setting of strategic games. First, we defined a Timed Strategic Game and then we have stated and proved a theorem that shows that, under certain circumstances, a Timed Nash Equilibrium is an "instantaneous" Nash Equilibrium. We also defined the concept of stable, unstable, partially left/right-handed stable Nash Equilibrium, which make easier the comprehension of the theory.

We explained the new timed game through some examples and adapted it to 4th Gen wars. The game can be applied to different fields such as value-based management, (e.g., value-based software engineering), earned value analysis, and share-market dynamics. Additionally, we think that simulation can be successfully applied to this field. The aim would be to try out the suitability of the application of the theory in a number of different situations and contexts, before applying it in the field. We leave all these points as future work.

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